Normal Modes as Eigenvectors of Coupled Oscillators in a Mechanical System

Syahrizal Bani Khairan - 13523063[1,2]
Program Studi Teknik Informatika
Sekolah Teknik Elektro dan Informatika
Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia
[1]13523063@std.stei.itb.ac.id, [2]syahrizal.khairan@gmail.com

***Abstract*— A mechanical system that comprises of coupled oscillators could have a complex and even nonperiodic motion. However, there exists a pattern in which the motion of such mechanical system can be completely described as the superposition of modes that oscillate at well-defined frequency. These normal modes correspond to the eigenvectors of the space of all displacements of each oscillator. This paper will demonstrate normal modes in a mechanical system consisting of rigid solid objects and springs, though the concept could generalize to other oscillating systems in other discipline of science.**

***Keywords*—coupled oscillators, normal modes.**

I. Introduction

An oscillation in physics is a physical quantity or a measure that varies periodically in time about a central value. Examples of Oscillators include swinging pendulum, electrical circuit of inductors and capacitors, and the vibrations of the atoms of a molecule.

In an oscillating system with more than one degree of freedom, the variables influence one another leading to a more complex behavior. The motion of such compound oscillator is harder to describe but nonetheless still possible. By resolving the system into normal modes, the behavior of the system becomes a linear superposition of these modes that is simpler and efficient.

This paper will try to demonstrate the decomposition of oscillating system into normal modes in simple mechanical systems. A generalized form that applies to other oscillating system will be shown.

II. Simple Harmonic Oscillator

An oscillation is a repetitive variation or fluctuation in a system about a central equilibrium position. It occurs when an object or quantity moves back and forth in a regular manner due to a restoring force that brings it back toward its equilibrium state. Oscillations can occur in various forms, such as mechanical vibrations in a pendulum, electrical oscillations in circuits, or even wave-like phenomena in fluids and light. The key characteristics of oscillations include their amplitude (the maximum extent of displacement from equilibrium), frequency (how many oscillations occur in a unit of time), and period (the time it takes for one complete oscillation). Oscillatory motion is fundamental in many physical systems and plays a crucial role in fields ranging from engineering and physics to biology and astronomy.

The simplest form of oscillation is a one-dimensional object undergoing simple harmonic motion (abbreviated SHM). An example of system that is modeled as a simple harmonic oscillator is that of a mass attached to a spring:

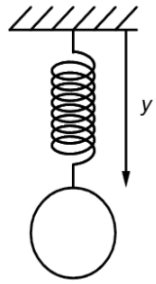


Figure 1. A mass suspended by a spring

Let y and y_0 be the length of the spring and the natural length of the spring, respectively. According to Hooke's Law, the force exerted on the mass is proportional to $y - y_0$. By Newton's second law, the equation of motion of the mass is

$$m\frac{d^2y}{dt^2} = -k(y - y_0) + mg.$$

This differential equation has solution of the form

$$y = y_0 + \frac{mg}{k} + A\sin(\omega t - \varphi).$$

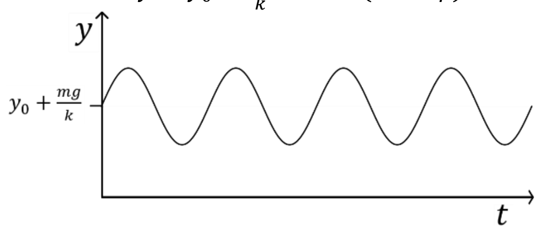


Figure 2. Graph of y with respect to time

The motion described by this equation is a periodic up-and-down motion of the mass. The constant A and φ are contants determined by the initial state of the system. Constant A physically represents the amplitude of the oscillation whereas φ is the initial phase angle.

Of particular interest is the constant ω, which is the angular frequency of the oscillation and related to the frequency by $f = \omega/2\pi$. This constant is not influenced by the initial state of the system, but rather the arrangement of the system itself. In this example, the value of ω is determined by

$$\omega = \sqrt{k/m}.$$

This angular frequency is thus inherent to the system itself and is its natural frequency.

Any generalized coordinate can also oscillate in a simple harmonic manner. Instead of the usual cartesian coordinate that denotes position, a generalized coordinate may describe the angle of rotation, length along a curve, or even a linear combination of other coordinates. A well-chosen set of coordinates that spans the whole space of the system may simplify the analysis of that system.

A system that can be modelled as a differential equation in generalized coordinate x of the form

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

undergoes simple harmonic motion with angular frequency ω and has equation of motion of the form

$$x = A sin(\omega t - \varphi).$$

Ideally, a more complex dynamical system such as coupled oscillators shall be simplified as a linear superposition of simple harmonic oscillators. Normal modes provide such simplification and lead to a deeper understanding of the behavior of the system.

III. Normal Modes

In a complex oscillating dynamical system that has two or more degrees of freedom, it is easier to analyze the system by transforming it into orthogonal coordinates that each independently oscillate at a single frequency. These component oscillations may be referred to as *normal modes.*

The most general motion of the system is a linear combination of its normal modes. This is akin to a transformation where the modes became the set of basis that span the full range of the system.

Normal modes is found by solving the eigenvalue problem that arises from the simultaneous differential equation. The frequencies at which the component oscillations vibrate are the eigenvalues. As will be demonstrated, the process of finding normal modes is similar to finding eigenvectors.

A. Example of Normal Mode

Sometimes the normal modes can be found easily by correctly identifying or spotting the set of coordinates that nicely describe the system. As an example, consider a system of a stick that is suspended from above by two massless springs attached to each end of the stick. The stick may move vertically or rotate about its center of mass. To simplify the problem, the oscillation is small relative to the length of the stick.

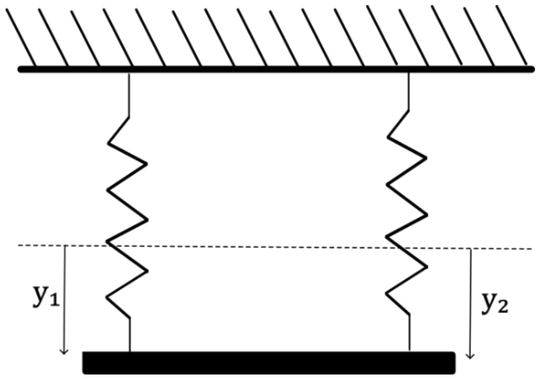


Figure 3. A stick hung on each end by two springs

The system has two degrees of freedom and thus two normal modes and two equations of motion. One may choose the differences of the length of each spring to its natural length to be the coordinates of the system referred as y_1 and y_2. To apply the physics laws, the coordinates must be transformed first.

The first equation comes from Newton's second law. The vertical position of the center of mass is half of the sum of the vertical positions of each end of the stick:

$$\frac{y_1 + y_2}{2} = y.$$

Newton's second law can be applied as if the system is a point mass at the center of mass:

$$m\frac{d^2}{dt^2}\left(\frac{y_1 + y_2}{2}\right) = mg - ky_1 - ky_2$$

$$\Leftrightarrow \frac{d^2}{dt^2}\left(\frac{y_1 + y_2}{2}\right) = g - \frac{2k}{m}\left(\frac{y_1 + y_2}{2}\right)$$

Let $y' = \frac{y_1+y_2}{2} - \frac{mg}{2k}$. The center of equilibrium shifts as denoted by the term $mg/2k$ due to the effect of gravity. The equation becomes

$$\frac{d^2y'}{dt^2} = \frac{2k}{m}y'.$$

The second equation comes from the Newton's second law for rotation: $I\, d^2\theta/dt^2 = \Sigma\tau$ with I being the moment of inertia of the stick. Let L be the length of the stick. By using small-angle approximation ($\sin\theta \approx \theta$ and $\cos\theta \approx 1$), the angle of rotation is given by

$$\frac{y_2 - y_1}{L} = \sin\theta \approx \theta$$

and thus

$$I\frac{d^2}{dt^2}\left(\frac{y_2 - y_1}{L}\right) = \frac{ky_2L}{2} - \frac{ky_1L}{2}$$

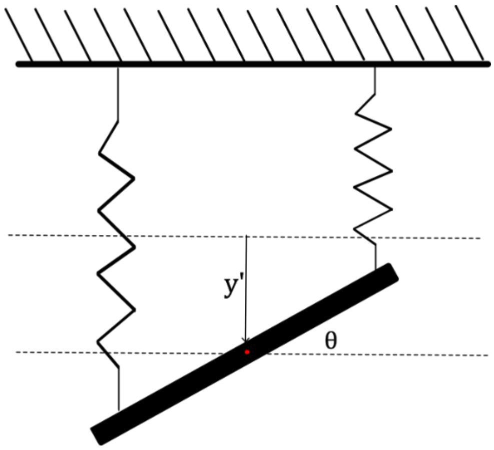


Figure 4. The same system now parametrized by different variables

The two equations can be simplified into the following set of equations:

$$\frac{d^2y'}{dt^2} = \frac{2k}{m}y',$$
$$\frac{d^2}{dt^2}\left(\frac{y_2 - y_1}{L}\right) = \frac{kL^2}{2I}\left(\frac{y_2 - y_1}{L}\right).$$

These differential equations are of the SHM differential equation form in terms of variables y' and $\frac{y_2-y_1}{L}$. Because these equations are independent of each other, they each denote a normal mode, and the variables are normal coordinates of the system. It turns out that these normal modes correspond to the constraint of the system, denoting the vertical motion of the center of mass and the rotational motion. In these new coordinates the equations of motion are as follow:

$$y' = Asin\left(\sqrt{\frac{2k}{m}}t - \alpha\right),$$
$$\frac{y_2 - y_1}{L} = Bsin\left(\sqrt{\frac{kL^2}{2I}}t - \beta\right).$$

These equations are clearer and capture the essential characteristics of the system. By substituting back y' and rearrangements, the equation of motion in the original coordinates are:

$$y_1 = \frac{1}{2}\left[\frac{mg}{k} + 2Asin\left(\sqrt{\frac{2k}{m}}t - \alpha\right) - LBsin\left(\sqrt{\frac{kL^2}{2I}}t - \beta\right)\right]$$
$$y_2 = \frac{1}{2}\left[\frac{mg}{k} + 2Asin\left(\sqrt{\frac{2k}{m}}t - \alpha\right) + LBsin\left(\sqrt{\frac{kL^2}{2I}}t - \beta\right)\right]$$

The motion of y_1 and y_2 is more complicated and the equation obscures the fact that it is just made up of two component that oscillate sinusoidally at pure frequency. The difference may be slight but resolving a more complex system into normal modes, one that has more degrees of freedom, will lend a better conceptual understanding and easier computation.

An oscillating system that has n degrees of freedom has n amount of normal modes. Such system has a behavior described by a n differential equation of the form

$$\frac{d^2}{dt^2}\vec{x} = V\vec{x}$$

Where $\vec{x}$ is a column vector denoting the coordinates. It is assumed the differential equation has a complete solution, and therefore the matrix V is non-singular. Matrix V then has the same number of eigenvalues as the dimension of the space, and hence the same number of normal modes.

In this example, normal modes are naturally found by trying to apply physical laws to describe the system. In other cases, normal modes may be found by a method. A procedural method to find the normal modes is shown in the following section.

B. Method of Finding Normal Modes in Coupled Oscillators

Let us consider a common and simplest form of coupled oscillators which is a system of two masses connected by three springs. Start with the equation of motion

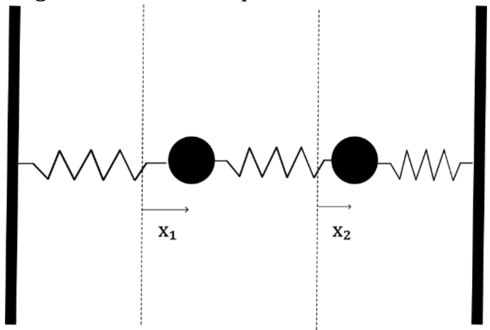


Figure 5. A system of two masses connected by three springs

$$m\frac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2,$$
$$m\frac{d^2x_2}{dt^2} = -kx_2 - k(x_2 - x_1) = -2kx_2 + kx_1,$$

which can be rearranged, with $\omega^2 = k/m$, into

$$\frac{d^2x_1}{dt^2} + 2\omega^2x_1 - \omega^2x_2 = 0,$$
$$\frac{d^2x_2}{dt^2} + 2\omega^2x_2 - \omega^2x_1 = 0.$$

With some rearrangement, the equations can be condensed into matrix form

$$\frac{d^2}{dt^2}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-2\omega^2 & \omega^2\\ \omega^2 & -2\omega^2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

The technique to solve the differential equation is by simply substituting the form

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}A_1\\A_2\end{pmatrix}\sin\alpha t.$$

Given that the second derivative of $\sin\alpha t$ is $-\alpha^2\sin\alpha t$, the equation becomes

$$-\alpha^2\sin\alpha t\begin{pmatrix}A_1\\A_2\end{pmatrix} = \sin\alpha t\begin{pmatrix}-2\omega^2 & \omega^2\\ \omega^2 & -2\omega^2\end{pmatrix}\begin{pmatrix}A_1\\A_2\end{pmatrix} \Longrightarrow$$
$$0 = \begin{pmatrix}-\alpha^2 + 2\omega^2 & -\omega^2\\ -\omega^2 & -\alpha^2 + 2\omega^2\end{pmatrix}\begin{pmatrix}A_1\\A_2\end{pmatrix}$$

To obtain nontrivial solution of A_1 and A_2, we must have the determinant of the matrix to equal zero. Notice that this is remarkably similar to the process of finding eigenvalues.

$$0 = \alpha^4 - 4\alpha^2\omega^2 + 3\omega^4.$$

The roots of this equation turn to be $\alpha = \pm\omega$ and $\alpha = \pm\sqrt{3}\omega$. If $\alpha = \pm\omega$, then $A = B$. If $\alpha = \pm\sqrt{3}\omega$ then $A = -B$. It is understood that each of these two cases are normal modes of the system. Breezing through the math here, the solution to the original system of differential equation is then the superposition

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega t) + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega t)$$
$$+ A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(\sqrt{3}\omega t) + A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(\sqrt{3}\omega t)$$

By combining sine function with the same frequency, we get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega t + \varphi_+)$$
$$+ B_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(\sqrt{3}\omega t + \varphi_-).$$

The normal mode $(1, 1)$, which oscillates at frequency ω, corresponds to the simultaneous back and forth motion of both masses ω. Normal mode $(1, -1)$, which oscillates at frequency $\sqrt{3}\omega$, corresponds to each mass moving back and forth in opposing direction. The original system is now decoupled into these two normal modes.

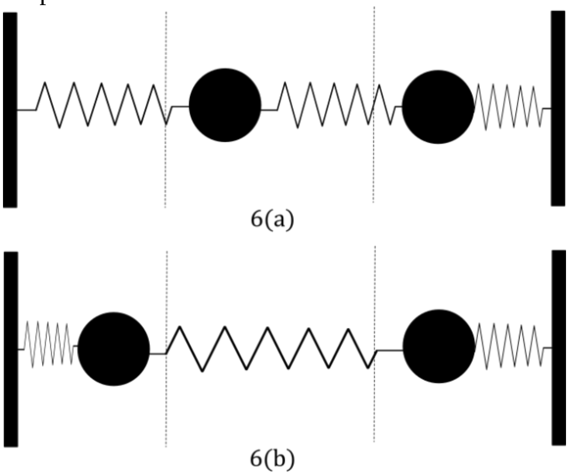


Figure 6. shows the two normal modes of the system. Figure 6(a) shows the masses oscillating simultaneously in the same diretion. Figure 6(b) shows the masses oscillating in opposing direction

As shown in Figure 6, the normal modes reveal some interesting motion. Figure 6(a) shows the system when it oscillates purely in normal mode $(1, 1)$. The masses appears to be moving back-and-forth as a single mass. In this mode, the spring in the center effectively does not exist. The frequency of this mode is $\sqrt{k/m}$. The system moves as if only influenced by one spring.

Figure 6(b) shows the other normal mode where each mass oscillates in opposite direction. Each spring now contributes more forces to the masses, which makes the masses oscillate faster.

C. Generalized System of N Coordinates

In the previous example, $(1, 1)$ and $(1, -1)$ are eigenvectors with the frequencies being their corresponding eigenvalue. Indeed, the process of finding normal modes is that of finding eigenvectors of the system coordinate space.

By describing the motion in terms of normal modes, a change of basis occurs from the original coordinates used to describe the system to the normal modes. Recall the matrix equation

$$\frac{d^2}{dt^2}\vec{x} = V\vec{x}.$$

This equation encodes a simultaneous differential equation that describe an oscillating system. A 2 dimensional example is shown in the previous example.

Generally, a coupled oscillating system that has n degrees of freedom is described by n equation:

$$\frac{d^2}{dt^2}x_1 = V_{11}x_1 + V_{12}x_2 + \cdots + V_{1n}x_n,$$
$$\frac{d^2}{dt^2}x_2 = V_{21}x_1 + V_{12}x_2 + \cdots + V_{2n}x_n,$$
$$\vdots$$
$$\frac{d^2}{dt^2}x_n = V_{n1}x_1 + V_{n2}x_2 + \cdots + V_{nn}x_n.$$

The matrix V is defined as thus.

If normal modes are used as the basis of the coordinate system, the system of equations become a system of n amount of independent simple harmonic differential equations as has been demonstrated before:

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} V_{11}' & 0 & \cdots & 0 \\ 0 & V_{22}' & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & V_{nn}' \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}.$$

One can achieve such transformation by diagonalizing the matrix V. Let us assume that V has n distinct eigenvalues. In this condition, V is diagonalizable.

Let $V = P^{-1}V'P$. Then

$$\frac{d^2}{dt^2}\vec{x} = P^{-1}V'P\vec{x} \Longrightarrow$$
$$P\frac{d^2}{dt^2}\vec{x} = PP^{-1}V'P\vec{x} \Longrightarrow$$
$$P\frac{d^2}{dt^2}\vec{x} = V'P\vec{x} \Longrightarrow$$
$$\frac{d^2}{dt^2}P\vec{x} = V'P\vec{x}.$$

The last step can be done because P has no dependece to time and differentiation is a linear operation. Notice that matrix P is naturally a change of basis matrix, with eigenvectors of matrix V as its columns

$$\vec{x}' = P\vec{x}.$$

Each column of P is an eigenvector that correspond to a normal mode. The diagonal elements of V' is proportiona; to the square of the frequency of each normal mode.

What has not been discussed much is the case when one of the eigenvalues happen to be zero or there happens to be multiple same eigenvalues. In this case, some of the normal modes will be degenerate. This just means that the dynamical system is made up of fewer fundamental oscillation than the amount of coordinates used to describe the system.

IV. Conclusion

The motion of a dynamical system of coupled oscillators can be complex. Describing the behavior of such system as being composed of independent oscillators lend to a deeper conceptual understanding of the system and efficient computation.

Though this paper only considers examples in mechanical physics, an oscillating system widely occurs in other disciplines of science. The generalized form of normal mode can be applied to any other oscillating dynamical system.

V. ACKNOWLEDGMENT

The author would like to express my deepest gratitude to Ir. Rila Mandala, M.Eng., Ph.D., for his invaluable guidance, support, and encouragement throughout the preparation of this paper. His expertise, insightful feedback, and dedication to teaching have greatly enhanced my understanding of the subject matter.

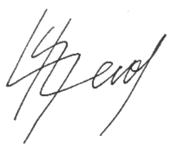
REFERENCES

[1] Goldstein, Herbert; Poole, Charles P.; Safko, John L. *Classical mechanics*. San Francisco, Munich: Addison Wesley, 2008.
[2] Morin, David. *Classical Mechanics: With Problems and Solutions.* Cambridge: Cambridge University Press, 2008.

PERNYATAAN

I hereby declare that this paper is an original work, and does not involve adaptation, translation, or plagiarism of any other individual's work.

Bandung, 2 Januari 2025



Syahrizal Bani Khairan - 13523063